An Introduction to Center Manifold Theory

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Abstract

One of the main methods of simplifying the study of dynamical systems is to reduce the dimensions of the system. Center Manifold Theory gives a mathematical prescription of carrying out this reduction near equilibria and provides information regarding the stability of equilibria. The present text is meant to be a basic introduction to the theory of Center Manifolds.

1 Introduction

In the study of dynamical systems, one often encounters parameters that appear in the governing equations of the system. As one varies these parameters, changes may occur in the qualitative structure of the equilibrium solutions. These changes are called *bifurcations* and the associated threshold values of the parameters are called *bifurcation values*. The linearized behavior of the system near an equilibrium point reveals a lot about the qualitative behavior of the system in the neighborhood (of the equilibrium point). In particular, the eigenvalues of the linear part of the governing equations help to determine whether the system behavior is stable or unstable. However, when a system loses stability, one usually observes that the number of eigenvalues and eigenvectors which are associated with this change is typically small. Thus, bifurcation problems often involve situations where the linearized system has a very large stable part and a smaller number of *critical modes* which change from stable to unstable as the bifurcation parameters cross the threshold values. The central tenet of bifurcation theory is that the dynamical behavior of the system near the onset of instability is dictated by the evolution of these critical modes, while the stable modes behave more passively. Center Manifold Theory (abbreviated as CMT henceforth) is a rigorous formulation of this observation and it usually allows one to reduce a large problem to a much smaller one. Thus CMT serves to reduce the dimension of the system under investigation and in this respect, the theory plays the same role for dynamic problems as the Lyapunov-Schmidt (-Koiter) analysis does for static solutions.

While the above description is useful for getting a broad picture, if one wishes to gain insight into the workings of CMT, one needs to be introduced to the three theorems that form the backbone of the theory. The first theorem establishes the existence of a center manifold and its smoothness properties. The second theorem guarantees that the stability of the reduced system on the center manifold determines the stability of the full system. Finally, the third theorem gives insight regarding the approximation of the center manifold.

The ideas behind invariant manifold theory have a long history. The first rigorous results on invariant manifolds were presented by Hadamard (1901), Lyapunov (1907) and Perron (1928). They proved the existence of stable and unstable manifolds for systems of differential equations and maps. Most modern texts usually follow the Lyapunov-Perron approach for obtaining the proofs of the theorems we outline in Section 3. The second of these theorems, commonly called the *Reduction Principle* today, originally appeared in the work of Pliss (1964). However, a complete proof of the theorem on the existence of center manifolds as well as

the complete proof of the Reduction Principle was obtained only some years later by Kelley (1967). The theory has since been extended by the work of Hale (1961), Hirsch et al. (1977), Smale (1963) and Carr (1981) to name a few. See Bates et al. (1998) for a wonderful review on the historical developments of the theory up till its present day form.
Before one delves into the heart of the theory, it is very useful to review some important results relating to dynamical systems. We are inclined to believe that the concepts involved in the CMT and its applications are non-trivial and so even a gentle introduction into the workings of this theory (such as the one this text is intended to provide) requires a knowledge of advanced concepts in the theory of dynamical systems. Therefore, in Section 2, we try to provide a good introduction to the relevant portions of the theory of dynamical systems that are repeatedly employed in discussing CMT. Section 3 discusses the Center Manifold Theorems and works through an example and Section 4 concludes the paper with a brief discussion of some of the applications and extensions of the theory.

2 Results from the Theory of Dynamical Systems

In this section we review some of the important results from the theory of Dynamical Systems without going through their proofs. The material in this section serves as the foundation for the material in the next section and much of it is adapted from Perko (2001) and Jordan and Smith (2007). Details of many of the proofs are included in the above references as well as Hirsch and Smale (1974). Motivated by the philosophy adopted in the seminal work Carr (1981), throughout this section as well as the next, we have adopted the simplest possible case of a finite dimensional autonomous system as our prototype. It seems reasonable to us that once this case is well understood, extension to other cases (such as continuous systems) involves simple, albeit technical modifications.

2.1 Linear Systems

The prototypical system that we will consider is a finite degree of freedom dynamical system whose dynamics is dictated by the following autonomous ordinary differential equation:

$$\dot{x} = f(x)\ ,\ x \in \mathbb{R}^n\ , f : \mathbb{R}^n \to \mathbb{R}^n\ , \tag{1}$$

with $x(0) = x_o$ as the initial condition. In general, the function $f(x)$ might be highly non-linear although it must satisfy certain smoothness properties. [1] Suppose that there is an equilibrium point for this system. It is instructive to study the linear behavior of the system in the vicinity of this equilibrium point. [2] This introduces the study of the linearized system:

$$\dot{x} = Ax,\ x(0) = x_0\ , \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$ is a matrix. One may ask the question whether there exists a solution to this linear system and further, if this solution is unique. This is adequately answered by the following result:

Theorem 2.1. ***(Fundamental Theorem For Linear Systems)*** *The linear initial value problem stated in (2) has a unique solution given by*

$$x(t) = e^{At}x_0\ . \tag{3}$$

[1] The fact that we have chosen an autonomous system is not a real restriction because, any non-autonomous system can be written as an autonomous one by letting $x \in \mathbb{R}^{n+1},\ x_{n+1} = t,\ \dot{x}_{n+1} = 1$. The theory does not change a lot.

[2] For certain kinds of systems, the correspondence of behavior of the linearized system to the non linear one can be exactly quantified. We introduce later the Hartman Grobman Theorem which makes this correspondence concrete and among other things, assures us that the study of the linearized system is a good place to start.

Note that by the term e^{At} , $t \in \mathbb{R}$ we mean the matrix exponential, which is the $n \times n$ matrix defined by:

$$e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!} \ . \tag{4}$$

The proof of the above theorem follows by direct substitution of the solution into the ODE to verify that it is indeed a solution. The uniqueness of the solution becomes a consequence of the observation that the matrices e^{-At} and A commute (that is, $e^{-At}A - A\, e^{-At} = 0$). The set of mappings $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ may be regarded as describing the motion of points $x_0 \in \mathbb{R}^n$ along the trajectories of (2) and this set of mappings is called the *flow of the linear system*.

The matrix exponential[3] is easy to compute if one has knowledge of the eigenvalues and eigenvectors of A. In general, the matrix may have both real and complex eigenvalues. First we consider the case that the eigenvalues are all distinct. Suppose that A is a $n \times n$ matrix which has k real eigenvalues λ_j and corresponding eigenvectors v_j for $j = 1, \ldots, k$. Furthermore, suppose that it has $(n-k)/2$ distinct pairs of complex eigenvalues $\lambda_j = a_j + ib_j$ and $\bar{\lambda}_j = a_j - ib_j$ and corresponding eigenvectors $w_j = u_j + iv_j$ and $\bar{w}_j = u_j - iv_j$ for $j = k+1, \ldots, (n+k)/2$. Then a theorem from linear algebra (Perko, 2001) states that the matrix

$$P = [v_1 \ \ldots \ v_k \ \ v_{k+1} \ \ u_{k+1} \ \ldots \ v_{\frac{n+k}{2}} \ \ u_{\frac{n+k}{2}}] \ , \tag{5}$$

is invertible and it block diagonalizes the matrix A, i.e.

$$P^{-1}AP = diag[\lambda_1, \ \ldots \ \lambda_k, \ B_{k+1}, \ \ldots \ , B_{\frac{n+k}{2}}] \ , \tag{6}$$

where $B_j = \begin{pmatrix} a_j & -b_j \\ b_j & a_j \end{pmatrix}$ for $j = k+1, \ \ldots \ , (n+k)/2$ are 2×2 blocks. Using (3) it follows that the solution to (2) in this case, is given by:

$$x(t) = P\ diag[e^{\lambda_1 t}, \ \ldots \ , e^{\lambda_k t}, \ R_{k+1}, \ \ldots \ , R_n]P^{-1}\, x_0 \ , \tag{7}$$

where $R_j = e^{a_j t} \begin{bmatrix} \cos(b_j t) & -\sin(b_j t) \\ \sin(b_j t) & \cos(b_j t) \end{bmatrix}$ for $j = k+1, \ \ldots \ , (n+k)/2$ are 2×2 blocks. Thus the solution to (2) is known explicitly. In case of repeated eigenvalues of A the above procedure needs to be modified. We need the following:

Definition 2.1.1. Let λ be an eigenvalue of the $n \times n$ matrix A of multiplicity $m \leq n$. Then for $k = 1, \ \ldots \ , m$, any nonzero solution v of $(A - \lambda I)^k v = 0$ is called a *generalized eigenvector* of A.

We call a matrix N *nilpotent of order* k if $N^r \neq 0$ when $r \in \mathbb{N}, 1 < r < k$ but $N^k = 0$. The following theorem (Hirsch and Smale, 1974) becomes important:

Theorem 2.2. *Suppose A is a real $n \times n$ matrix with real eigenvalues $\lambda_1, \ \ldots \ , \lambda_n$ repeated according to their multiplicity. Then there exists a basis of generalized eigenvectors of $\mathbb{R}^n$. If $v_1, \ \ldots \ , v_n$ is any basis of generalized eigenvectors for $\mathbb{R}^n$, the matrix $P = [v_1, \ \ldots \ , v_n]$ is invertible, $A = S + N$, where $P^{-1}SP = diag[\lambda_j]$ $(j = 1, \ \ldots \ n)$, the matrix $N = A - S$ is nilpotent of order $k \leq n$ and the matrices S and N commute.*

The theorem above immediately implies that the solution to (2) must be of the form:

$$x(t) = P\ diag[e^{\lambda_j t}]P^{-1}\big[I + Nt + \ \ldots \ + \frac{N^{k-1}t^{k-1}}{(k-1)!}\big]\ x_0 \ . \tag{8}$$

Thus, once again the solution to (2) is known explicitly. The results in (7) and (8) can be easily extended to other cases, such as the case in which the matrix A has repeated real as well as complex eigenvalues and results similar to the ones above can be obtained (Perko, 2001; Hirsch and Smale, 1974).

[3] It can be shown using the Weierstrass M-Test that the matrix exponential defined this way is absolutely and uniformly convergent under certain general hypothesis. More details can be found in Perko (2001) and Rudin (1976).

2.2 Linear Stability Theory

We begin this section by the introducing some definitions that become important in discussing the stability properties of the linear system (2). Let $w_j = u_j + iv_j$ be a generalized eigenvector corresponding to the eigenvalue $\lambda_j = a_j + ib_j$. Then:

Definition 2.2.1. $E^s = Span\{u_j, v_j : a_j < 0\}$ is called the *stable subspace*, $E^c = Span\{u_j, v_j : a_j = 0\}$ is called the *center subspace* while $E^u = Span\{u_j, v_j : a_j > 0\}$ is called the *unstable subspace*. Thus, the spaces E^s, E^c and E^u are the subspaces of $\mathbb{R}^n$ spanned by the real and imaginary parts of the generalized eigenvectors w_j corresponding to eigenvalues λ_j which have negative, zero and positive real parts respectively.

Based on Theorem 2.2, it immediately follows that $\mathbb{R}^n = E^s \oplus E^c \oplus E^u$.

Definition 2.2.2. If all eigenvalues of the matrix A have non-zero real part, then the flow given by $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ is termed *hyperbolic* and (2) is called a *hyperbolic linear system*.

Definition 2.2.3. A subspace $E \subset \mathbb{R}^n$ is said to be *invariant with respect to the flow* $e^{At} : \mathbb{R}^n \to \mathbb{R}^n$ if $e^{At}E \subset E \ \ \forall t \in \mathbb{R}$.

From these ideas stems out the very important result that any solution of the system starting in E^s, E^c or E^u at the time $t = 0$ remains in E^s, E^c or E^u respectively, for any time (Perko, 2001). In particular, if the solution started in the stable subspace, it approaches the origin exponentially fast regardless of the form of A. On a general note it also follows that any solution to the linear system remains bounded in norm as $t \to \infty$ if and only if all the eigenvalues of A have non-positive real part. [4] More precisely we have:

Theorem 2.3. *The subspaces E^s, E^c and E^u defined as above are invariant with respect to the flow e^{At} of (2). In the particular case of $x_0 \in E^s$ one has that $\lim_{t \to \infty} e^{At}x_0 = 0$. Further if all eigenvalues of A have a negative real part then there are positive constants M, c such that $\forall x_0 \in \mathbb{R}^n$ it holds that $|x(t) = e^{At}x_0| \leq Me^{-ct}|x_0|$.*

The above theorem tells us everything about the asymptotic behavior of the solutions of the linear system. With the tools of the linear system in hand, we can now study the behavior of a nonlinear system qualitatively.

2.3 Local Theory of Nonlinear Systems

We return to the prototypical autonomous ODE, $\dot{x} = f(x)$ in this section. We are mainly interested in the case where f is a continuously differentiable function over an open set $E \subset \mathbb{R}^n$. [5] Suppose now that x_0 is an equilibrium point for the system, i.e. $f(x_0) = 0$. One quantity of interest in further analysis is the matrix $A = Df(x_0)$, i.e. the derivative of f at x_0. We would call x_0 a *hyperbolic equilibrium point* if all the eigenvalues of $A = Df(x_0)$ have non zero real part. One may begin by asking the questions of existence and uniqueness and in this case, we have the following result (Perko, 2001; Hartman, 1982)

Theorem 2.4. ***(Fundamental Existence-Uniqueness Theorem)*** *Let E be an open subset of $\mathbb{R}^n$ containing x_0 and suppose that f is continuously differentiable on E. Then there exists an interval $[-a, a] \subset \mathbb{R}$ such that the initial value problem $\dot{x} = f(x)$, $x(0) = x_0$ has a unique solution $x(t)$ on the interval.*

[4] This is commonly called the linear stability criterion.

[5] See Rudin (1976) for the definition of the derivative of a function which maps vectors to vectors, such as the function f in this case.

The proof of the above theorem is based on setting up a set of successive approximations to the solution (called the *Picard Iterates*, see for example: Hartman (1982)) and then using the Fixed Point Theorem (Rudin, 1976) to show that these approximations in fact converge to the actual solution locally. The uniqueness of the fixed point ensures local uniqueness. The smoothness hypothesis on f becomes important in establishing the global existence result. One may next ask how sensitive the solution is with respect to initial conditions and how any parameters that f may be a function of (i.e. $\dot{x} = f(x, \mu)$, $\mu \in \mathbb{R}^p$) influence the solution. Both these questions become very important in practical problems, for instance in the context of bifurcating systems. In fact, these questions are related to issues of global uniqueness of the solutions. We will not explicitly answer these questions here. However, we would like to point out that there is a well developed body of theory related to these questions[6] and roughly speaking they say that the solution $x(t, x_0, \mu)$ depends continuously on the initial conditions x_0 as well as the parameters μ. Further, the degree of continuity of the solution is the same as the degree of continuity of f. In fact, there is a *maximal interval of existence* of solution over which a unique solution to the initial value problem exists.
Analogous to the development of the linear theory one can define the *flow* of a non-linear system. This is done as follows (Perko, 2001):

Definition 2.3.1. Let E be an open subset of $\mathbb{R}^n$ and let f be continuously differentiable on E. For $x_0 \in E$, let $x(t, x_0)$ be the solution of the initial value problem (1) defined on its maximal interval of existence $I(x_0)$. Then $\forall t \in I(x_0)$ the set of mappings ϕ_t defined by $\phi_t = x(t, x_0)$ is called the *flow of the differential equation*.[7] A set S is *invariant with respect to the flow* if $\phi_t(S) \subset S$, $\forall t \in \mathbb{R}$.

With the above theorems and definitions in place, we are now in a position to introduce two results which are of immense significance in the study of non-linear systems: The Hartman-Grobman Theorem and the Stable Manifold Theorem. We will first discuss the Hartman-Grobman theorem. For doing so, we need to introduce the concept of *topologically equivalent spaces*. First, we recall that a *homeomorphism* (also called a *toplogical isomorphism*) between two sets A and B is a continuous bijection (i.e. a map that is one to one and onto, (Rudin, 1976)) whose inverse is also continuous. We say that two sets (or spaces) are *toplogically equivalent* if there is a homeomorphism between the two sets. Intuitively, this means that we say two sets have similar qualitative properties if we can continuously deform one into the other.[8] The Hartman-Grobman theorem (Hartman, 1982; Perko, 2001) then states:

Theorem 2.5. ***(Hartman-Grobman Theorem)*** *Let E be an open subset of $\mathbb{R}^n$ and let f be continuously differentiable on E. Suppose E contains a point x_0 which is a hyperbolic equilibrium point of the ODE $\dot{x} = f(x)$. Then there exists a homeomorphism H of an open set U containing x_0, onto an open set V containing x_0 such that for any point $y_0 \in U$, there is an open interval $I_0 \subset \mathbb{R}$ containing zero such that for all $y_0 \in U$, $t \in I_0$ we have $H \circ \phi_t(y_0) = e^{At} H(y_0)$.*[9]

Informally, the theorem says that in the vicinity of a hyperbolic equilibrium point, the nonlinear system $\dot{x} = f(x)$ has the same qualitative features as does the linear system $\dot{x} = Ax$. More specifically, the theorem says that there is a homeomorphism that maps trajectories of the non-linear system near the equilibrium point onto trajectories of the linear system near the equilibrium point and preserves the parametrization by time while doing so. Thus, this theorem gives us confidence that a study of the linearized version of the nonlinear system near an equilibrium point yields meaningful results relevant to the actual nonlinear system.
Next we will discuss the Stable Manifold Theorem. In many ways, this theorem serves as the direct precursor

[6] Most of these important results can be found in Gronwall (1919); Perko (2001); Jordan and Smith (2007).

[7] The reader may observe that the concept of the flow map is basically just the introduction of the notation $\phi_t(x_0) = x(t, x_0)$. For a linear system, $\phi_t = e^{At}$.

[8] For example if two solutions of an equation are $y = x^2$ and $y = x^2 + c_1$, then they are both parabolic solutions and they are topologically equivalent. One can deform one solution into the other, in this case by a shift along the axes.

[9] Here $\phi_t(y_0)$ is the flow corresponding to the ODE $\dot{x} = f(x)$.

of the Center Manifold Theorem that discussed in the next section. We begin by recalling the concept of a manifold in a loose sense (Kreyszig, 1991): A *manifold* is a (topological) space, such that on a small enough scale it resembles the Euclidean space of a certain dimension, called the *dimension of the manifold*. Thus, while a line and a circle are one-dimensional manifolds, a plane and the surface of a ball are two-dimensional manifolds. A *differentiable manifold* is a manifold that is locally similar enough to Euclidean space to allow one to do calculus on the manifold. We now state the Stable Manifold Theorem (Perko, 2001):

Theorem 2.6. ***(Stable Manifold Theorem)*** *Let E be an open subset of $\mathbb{R}^n$, let f be continuously differentiable on E and let $\phi_t(y)$ be the flow of the nonlinear system $\dot{x} = f(x)$. Suppose that x_0 is a hyperbolic equilibrium point such that $A = Df(x_0)$ has k eigenvalues with negative real part and $n-k$ eigenvalues with positive real part. Then there exists a k-dimensional differentiable manifold W^s tangent to the stable subspace E^s of the linear system $\dot{x} = Ax$ at x_0 with the properties that $\forall t \geq 0,\ \phi_t(W^s) \subset W^s$ and that for any $y \in W^s$ it holds that $\lim_{t\to\infty} \phi_t(y) = x_0$. Furthermore, there is also an $n-k$ dimensional differentiable manifold W^u tangent to the unstable subspace E^u of the linear system at x_0 with the properties that $\forall t \leq 0,\ \phi_t(W^u) \subset W^u$ and that for any $y \in W^u$ it holds that $\lim_{t\to-\infty} \phi_t(y) = x_0$.*

Qualitatively speaking, the theorem says that near a hyperbolic equilibrium point, the nonlinear system has stable and unstable (differentiable) manifolds W^s and W^u tangent to the stable and unstable subspaces E^s and E^u respectively, of the linearized system. Further, these manifolds W^s and W^u have the same dimension as E^s and E^u respectively. The theorem also says that the flow of the non-linear system leaves the stable manifold invariant for $t \geq 0$ while it leaves the unstable manifold invariant for $t \leq 0$. Finally, any point on the stable manifold approaches the equilibrium point x_0 as $t \to \infty$ while any point on the unstable manifold approaches the equilibrium point as $t \to -\infty$. Thus, points starting on the stable manifold continue to stay on it and approach the equilibrium arbitrarily close as time increases.

3 Center Manifold Theorems

We are now ready to begin our discussion of the theorems that constitute the framework of CMT, in their simplest form. The proofs of all these theorems depend on employing the Fixed Point Theorem in an essential way. Most of the material in this section is based on Carr (1981) as well as Perko (2001), Bressan (2003) and Guckenheimer and Holmes (1985).

3.1 Existence of the Center Manifold

Broadly speaking, the theorem that we are going to discuss in this subsection relaxes the assumption of hyberbolicity in the Stable Manifold Theorem (Theorem 2.6) and considers the case of the matrix $Df(x_0)$ having eigenvalues whose real parts might be positive, negative or zero. This immediately raises the technical issues of non-uniqueness and loss of smoothness that were not there in case of the Stable Manifold Theorem. However, in this case too, associated with the respective invariant subspace E^s, E^c and E^u are differentiable manifolds which are locally invariant under the flow of the ODE. Further, these three manifolds W^s, W^c and W^u are all tangent to their respective subspaces at x_0 and they have the same dimension as their respective subspaces. The center manifold method then focuses only on the center subspace and it effectively isolates the complicated asymptotic behavior associated with the original system. We now state the existence theorem following Bressan (2003) and Guckenheimer and Holmes (1985) and adopt the notation therein:

Theorem 3.1. ***(Center Manifold Theorem)*** *Let f be a C^r vector field on $\mathbb{R}^n$ (i.e., it is continuous up to it's r^{th} derivative) which vanishes at the origin*[10]*,i.e. $f(0) = 0$ and let $A = Df(0)$. Let the stable, center and*

[10]Here the origin has been shifted to the equilibrium point.

unstable invariant subspaces associated with be E^s, E^u and E^c respectively. Then there exists C^r stable and unstable invariant manifolds W^s and W^u tangent to E^s and E^u at 0 and a C^{r-1} center manifold W^c tangent to E^c at 0. The manifolds W^s, W^c and W^u are invariant for the flow of f. Furthermore, the stable and unstable manifolds are unique but the center manifold need not be.

One might naively guess that a simpler alternative to using the center manifold thorem for a system would be to project the system onto the linear subspace spanned by E^c. Thus, one would hope that near equilibrium, the projection of f onto E^c provides the correct qualitative picture of the dynamics in the center directions. However, there are counter examples to this assertion (Guckenheimer and Holmes, 1985) and the actual picture is provided by the tangent (center) manifold.
The Center Manifold Theorem really implies that the bifurcating system is locally topologically equivalent to the system of equations:

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) \ , \ \dot{\tilde{y}} = -\tilde{y} \ , \ \dot{\tilde{z}} = \tilde{z} \ \ , \ (\tilde{x}, \tilde{y}, \tilde{z}) \in W^c \times W^s \times W^u \ , \tag{9}$$

at the bifurcation point. We next need a systematic procedure for computing the reduced vector field $\tilde{f}$.

3.2 Reduced Dynamics on the Center Manifold

For the sake of simplicity and because it is the most important case physically we assume that the unstable manifold is empty[11]. Suppose further that the linear part of the (n-dimensional) system is in the following block diagonal form:

$$\dot{x} = Bx + f(x, y) \ , \ \dot{y} = Cy + g(x, y) \ , \ (x, y) \in \mathbb{R}^p \times \mathbb{R}^q \ , \ p + q = n \ , \tag{10}$$

where B and C are $p \times p$ and $q \times q$ matrices whose eigenvalues have, respectively, zero real parts and negative real parts, and the functions f and g as well as their first partial derivatives are 0 at the origin. Of course, if f and g are identically zero in (10) then there are two obvious invariant manifolds: $x = 0$ and $y = 0$, with the former being the stable manifold.
By definition, the center manifold is tangent to E^c (the $y = 0$ space) and we can represent it as the graph:

$$W^c = \{(x, y) : y = h(x)\} \ , \ h(0) = Dh(0) = 0 \ , \tag{11}$$

where $h : U \to \mathbb{R}^q$ is defined on some neighborhood $U \subset \mathbb{R}^p$ of the origin. We then consider the projection of the vector field on $y = h(x)$ onto E^c:

$$\dot{x} = Bx + f(x, h(x)) \ . \tag{12}$$

But since $h(x)$ is tangent to $y = 0$, the solutions to the above equation (12) provide a good approximation of the flow of $\dot{\tilde{x}} = \tilde{f}(\tilde{x})$ restricted to W^c. In fact one can prove rigorously (Pliss, 1964; Carr, 1981; Guckenheimer and Holmes, 1985) that:

Theorem 3.2. ***(Reduction Principle):*** *If the origin $x = 0$ of (12) is locally asymptotically stable (unstable) then the origin of (10) is also locally asymptotically stable (unstable).*

Thus the stability properties of (10) can be studied by looking at the stability properties of (12). It remains therefore, to find a faithful way of calculating $h(x)$. This is done next.

[11] Presence of a non-empty unstable manifold corresponds to a system which is unstable even by linear analysis. One can also proceed without making this assumption but the system of equations in (9) becomes more complicated.

3.3 Approximation of the Center Manifold

We now discuss how one can go about calculating $h(x)$ or at least approximating it. We substitute $y = h(x)$ in the second equation in (10) and use the chain rule to get:

$$\dot{y} = Dh(x)\dot{x} = Dh(x)[Bx + f(x, h(x))] = Ch(x) + g(x, h(x)) \ . \tag{13}$$

Clearly, this expresses the equation of the manifold:

$$N(h(x)) \equiv Dh(x)[Bx + f(x, h(x))] - Ch(x) - g(x, h(x)) = 0 \ , \tag{14}$$

with boundary conditions $h(0) = Dh(0) = 0$. This partial differential equation in h cannot be solved explicitly in general. The solution can be well approximated by a Taylor expansion about $x = 0$ as shown by the following theorem (Carr, 1981; Guckenheimer and Holmes, 1985):

Theorem 3.3. *If a function $\phi(x)$ with $\phi(0) = D\phi(0) = 0$ can be found such that $N(\phi(x)) = O(|x|^m)$ for some $m > 1$ as $|x| \to 0$ then it holds that $h(x) = \phi(x) + O(|x|^m)$ as $|x| \to 0$.*

Thus, this theorem tells us that we can approximate $h(x)$ as closely as we wish by employing series solutions of (14). However, it might be that such a series expansion is infact invalid, since in certain cases, W^c may not be analytic at the origin. Together, the theorems (3.1),(3.2) and (3.3) completely demonstrate how to reduce a problem onto the center manifold and obtain qualitative behavior regarding the stability. To see the ideas in this section in action, we work now through an example (Perko, 2001):

Example: Consider the nonlinear ODE system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} + \begin{bmatrix} x_1 y - x_1 x_2^2 \\ x_2 y - x_2 x_1^2 \\ x_1^2 + x_2^2 \end{bmatrix} \ . \tag{15}$$

Clearly $x_1 = x_2 = y = 0$ is an equilibrium point. We observe further that the ODE is already cast into a form where it's linear part and non-linearities have been separated out. We also notice that ODE for the variable y is in the stable part of the system since the associated linear part is $-y$. Now, the eigenvalues of the linear part corresponding to x_1 and x_2 are have zero real part (in fact, they are identically zero). So we conclude that the ODE's in x_1 and x_2 have to do with the center part of the system. From the discussion in this section, we are assured of the existence of a center manifold (Theorem 3.1) and further, the discussion in subsection 3.2 allows us to express the stable variable y as a function of the center variables x_1 and x_2. Thus, setting $x = (x_1, x_2)^T$ and $y = h(x)$, we have by equation (14), $Dh(x)[Bx + f(x, h(x))] - Ch(x) - g(x, h(x)) = 0$ and $h(0) = Dh(0) = 0$. Now, to work out a power series solution for h, we begin with a second order approximation and realize that we would need to set $h(x) = ax_1^2 + bx_1x_2 + cx_2^2 + O(|x|^3)$, so as to satisfy the conditions $h(0) = Dh(0) = 0$. Thus, we have:

$$\begin{aligned} B &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \ , \\ C &= [-1] \ , \\ f(x, h(x)) &= \begin{pmatrix} x_1 h(x) - x_1 x_2^2 \\ x_2 h(x) - x_2 x_1^2 \end{pmatrix} \ , \\ g(x, h(x)) &= x_1^2 + x_2^2 \ , \\ Dh(x) = (\frac{\partial h}{\partial x_1}, \frac{\partial h}{\partial x_2}) &= (2ax_1 + bx_2, bx_1 + 2cx_2) + O(|x|^2) \ . \end{aligned} \tag{16}$$

Substituting the above expressions into (14), and equating to zero, we get $a = 1, b = 0, c = 1$. Thus, we conclude that up to a second order approximation the center manifold is governed by the equation $h(x_1, x_2) = x_1^2 + x_2^2 + O(|x|^3)$. Hence, by equation (12), near the origin, the dynamics on the center manifold is given by $\dot{x}_1 = x_1^3 + O(|x|^4)$, $\dot{x}_2 = x_2^3 + O(|x|^4)$. The presence of the positive coefficients on the right hand side of the above uncoupled system indicates instability. Thus, this analysis clearly implies that the origin is a type of topological saddle that is unstable. However, if we were to naively use the center subspace approximation for the local center manifold, i.e., if we were to set $y = 0$ in the differential equations for x_1 and x_2, we would obtain $\dot{x}_1 = -x_1 x_2^2$, $\dot{x}_2 = -x_2 x_1^2$ and we would arrive at the wrong conclusion that the origin is a stable, nonisolated critical point for the system.

4 Applications and Extensions

First we apply the theory from the previous section to studying bifurcations, which probably form the biggest application of the theory of Center Manifolds. We then look at extensions/generalizations of the theory. Most of the material of this section has been adapted from Carr (1981), Marsden and McCracken (1976), Guckenheimer and Holmes (1985) and Bates et al. (1998).

4.1 Application to Bifurcation Theory

To explicitly see how the theory presented in the earlier section can be used to study bifurcations, we need to deal with parametrized families of systems. Consider the system of ODEs dependent on a vector parameter $\mu \in \mathbb{R}^k$

$$\dot{w} = F(w, \mu)\ ,\ F(0, \mu) = 0\ ,\ w \in \mathbb{R}^{p+q}\ ,\ p + q = n\ . \tag{17}$$

We say the $\mu = 0$ is a bifurcation point for (17) if the qualitative nature of the flow changes at $\mu = 0$,i.e. in the neighborhood of $\mu = 0$ topological properties of flow are different for different values of μ. Suppose that we linearize (17) about $w = 0$ and we get:

$$\dot{w} = C(\mu)w\ . \tag{18}$$

Clearly, if the eigenvalues of $C(0)$ all have non-zero real parts then the Hartman-Grobman Theorem (2.5) tells us that the solutions of (17) behave like the solutions of (18) for small $|\mu|$ and so $\mu = 0$ is not a bifurcation point. Thus, the situation is interesting from the local point of view only when $C(0)$ has eigenvalues with zero real parts. Now, suppose that $C(0)$ has p eigenvalues with zero real parts and q eigenvalues whose real parts are negative. As before, we are not interested in any positive eigenvalues as we want to investigate bifurcation of stable phenomena only. Thus we can write:

$$\dot{x} = A_\mu x + f_\mu(x, y)\ ,\ \dot{y} = B_\mu y + g_\mu(x, y)\ ,\ \dot{\mu} = 0\ ,\ (x, y) \in \mathbb{R}^p \times \mathbb{R}^q\ ,\ p + q = n\ . \tag{19}$$

where A_μ and B_μ are $p \times p$ and $q \times q$ matrices whose eigenvalues have, respectively, zero real parts and negative real parts and $x \in \mathbb{R}^p$, $y \in \mathbb{R}^q$. Further the functions f_μ , g_μ as well as their first partial derivatives vanish at $(x, y, \mu) = (0, 0, 0)$. Now by Theorem 3.1, the above system (17) has a $(p+k)$ dimensional center manifold $y = h(x, \mu)$. tangent to the (x, μ) space. Further, by Theorem 3.2, the behavior of small solutions of (19) is governed by the equations:

$$\dot{u} = Au + f(u, h(u, \mu), \mu)\ ,\ \dot{\mu} = 0\ . \tag{20}$$

In practical applications, very frequently p is 1 or 2 because the number of critical modes are small. Thus, the above procedure results in a tremendous reduction and we may then follow the local evolution of bifurcating solutions in the center manifold. We need to mention however, that the above analysis is local.

4.2 Extensions and Generalizations

The theory of center manifolds presented here is in its simplest possible form. Even in this form, the theory has tremendous use in studying problems related to bifurcation as well as problems in singular perturbation theory. However, there are several extensions and generalizations of the theory and we briefly mention some of them here:

1. One does not really need to have an equilibrium point in the theory developed above. All that is really needed is a fixed point. As such, there is a center manifold theorem for *diffeomorphisms* (a smooth invertible function between two differential manifolds, whose inverse is also smooth) at a fixed point. At such a fixed point, one can show the existence of invariant manifolds corresponding to the eigenspaces of the linearization of diffeomorphism about the fixed point, depending on whether the eigenvalue lies on, inside or outside the unit circle (Marsden and McCracken, 1976).

2. Center Manifold theory can be extended to study the global properties of nonlinear systems in the context of *maps*. Maps arise naturally in studying periodic solutions of differential equations. The extended form of the theory can, for example, be used to study bifurcation of maps (Carr, 1981).

3. We have presented the theory in a finite dimensional space, but one can develop an analogous theory for infinite dimensional systems (Carr, 1981). The basic setting then becomes a *Banach Space* (which is a complete, normed space). Matrices become *linear operators* in the infinite dimensional setting and for the purpose of subsequent developments, it turns out that one needs to use *semigroup theory*. The main motivation of introducing these technicalities into the theory is that being able to deal with infinite dimensional problems really means that one can study the behavior of solutions of partial differential equations as infinite dimensional ordinary differential equations. This program has been applied with good success in a lot of situations (Carr, 1981; Bates et al., 1998) including problems in elasticity theory (Ball, 1973).

4. Suppose that we have successfully applied the center manifold theorem to a system and so we are just concerned with the flow on the center manifold. It turns out that introducing additional coordinate transformations to simplify the analytic expression of the vector field on the center manifold is often fruitful. The resulting vector fields are called *normal forms* and the analysis of their dynamics yields a qualitative picture of the flows of each bifurcation type. For some systems, the normal form (truncated at a given degree) is even simple enough to become solvable and in this case one can ask whether this solution gives rise to a good asymptotic approximation to a solution of the original equation. These are some of the issues that are studied in the Normal Forms Theroy. More details can be found in Guckenheimer and Holmes (1985) and Perko (2001) and references therein.

5. For the case of dissipative systems, the concept of an *Inertial Manifold* introduced by Foias et al. (1988) is found to be very useful. This concept was originally developed by the authors for nonlinear evolutionary systems governed by ordinary and partial differential equations. These manifolds, are found to be an appropriate tool for the study of questions related to the long-time behavior of solutions of the evolutionary equations. The inertial manifolds contain the *global attractor*, they exponentially attract all solutions, and they are stable with respect to perturbations. The authors also showed in their paper, that in the infinite dimensional case this tool allows for the reduction of the dynamics to a finite dimensional ordinary differential equation.

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